Date: 13.03.2013
Teacher: Ceren Özbay
Number of Students: 16
Grade Level: 11I / IB SL
Time Frame: 45 minutes

Mathematical Induction

1. Goal(s)
   - To develop an understanding of process of mathematical induction.

2A. Specific Objectives (measurable)
   - Students will be able to use induction to prove integer sum formulas.

2B. Ministry of National Education (MoNE) Objectives
   - İD.11.1.2. Açık Önermeler ve İspat Teknikleri, İD.11.1.2.5. Tümevarım yöntemi ile ispat yapar.

2C. NCTM-CCSS-IB or IGCSE Standards:
   - The aim of this topic is to provide the opportunity for students to engage in logical reasoning, algorithmic thinking and applications. (IB- HL)

3. Rationale
   - Mathematical induction is required for reasoning about objects or events containing repetition, e.g. computer programs with recursion or iteration, electronic circuits with feedback loops or parameterized components. Thus automating mathematical induction is a key enabling technology for the use of formal methods in information technology. Failure to automate inductive reasoning is one of the major obstacles to the widespread use of formal methods in industrial hardware and software development.

4. Materials
   - Board.
   - At least two different colored board markers.
   - One worksheet to each student.
   - Projector
   - TI calculator
   - Computer

5. Resources
   - Haese &Harris Publications. Mathematics for the international student Mathematics HL –core book. The authors are Paul Urban, John Owen, David Martin, Robert Haese, Sandra Haese, and Mark Bruce.
   - IB sample exam papers related to the topic.

6. Getting Ready for the Lesson (Preparation Information)
   - Teacher should make sure that she gets the worksheets.
   - Teacher will check the computer.
   - Teacher will be sure that board markers are working.
   - Teacher will remind students they are supposed the do the rest of questions on the worksheet as homework.

7. Prior Background Knowledge (Prerequisite Skills)

- Students should know about the basic arithmetic operations.

Lesson Procedures

Transition: Good morning class! Today, we are going to learn mathematical induction.

8A. Engage (5 minutes)

- Write sum of first 100 positive integers on the board and ask students how they find sum of first 100 positive integers without using formula quickly.
- Say that students can probably notice that adding together many numbers can be tedious, unless they use a calculator. Before people had computers and calculators, they often searched for ways to make calculations easier. For example, logarithms, which students learned about logarithms, were developed to simplify calculations that involve large numbers. Over time, mathematicians have developed formulas that allow us to simplify the calculations of large sums.
- Tell the story about Gauss:

  In fact, German mathematician C.F. Gauss is often credited with discovering such a formula when he was a young child. The story is likely apocryphal (a legend), but it has been passed down since Gauss lived in the 1700's. According to the story, Gauss's teacher wanted to occupy students by having them add up large sets of numbers when he was at primary school. When Gauss was asked to add up the first 100 integers, he found the sum very quickly, by pairing the numbers:

  All of the numbers in the sum could be paired to make groups of 101. There are one hundred numbers being added, so there are such fifty pairs. Therefore the sum is 50(101) = 5050.The method Gauss used to solve this problem is the basis for a formula that allows us to add together the first n positive integers. However, in order to prove that the formula always works, we need to show that it works for all positive integers. In this lesson you will learn about mathematical induction, a method of proof that will allow you to prove that a particular statement is true for all positive integers. First we will present the method, and then we will prove Gauss's formula, as well another related sum.

Transition: Let's make a guess at a formula that will give us the sum of all the positive integers from 1 to n for any positive integer n.

B. Explore (10 min.)

- Let students write on their notebooks.
- Students will try to find the formula.
- Walk around and ask "how did you get the formula?" and help students to explore the formula.
- Check the students whether they work on or not.
- If somebody finds the formula, let her /his write it on the board.
- As an answer we can see a general form: there were 100 numbers, hence 50 pairs. So if there were n numbers, there would be $(n/2)$ pairs. The first and last numbers were 1 and 100. They added together to give us 101. This number was the sum of each pair in the overall sum. So in general, we could add together 1 and n to get the sum of each pair. Therefore we might hypothesize that the sum of the first n positive integers is $n((1 + n)/2)$.

Transition: How can we prove that this formula works for all positive integers n?

C. Explain (10 min.)

- Let students to make a guess.
- Say that mathematical induction will allow us to prove the formula.
- Then write the formula as follows:

Proposition notation

We use P_n to represent a proposition which is defined for every integer a where $n \geqslant a$.

For example, in the case of **Example 1**, our proposition P_n is

$$\text{“}\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}.....+\frac{1}{n(n+1)}=\frac{n}{n+1} \quad \text{for} \quad n\in Z^+\text{”}$$

Notice that P_1 is "$\frac{1}{1\times 2}=\frac{1}{2}$" and P_2 is "$\frac{1}{1\times 2}+\frac{1}{2\times 3}=\frac{2}{3}$"

and P_k is "$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+.....+\frac{1}{k(k+1)}=\frac{k}{k+1}$".

THE PRINCIPLE OF MATHEMATICAL INDUCTION

Suppose P_n is a proposition which is defined for every integer $n \geqslant a$, $a \in Z$.

Now if
- P_a is true, and
- P_{k+1} is true whenever P_k is true,

then P_n is true for all $n \geqslant a$.

This means that for $a = 1$, say, and the two above conditions hold, then the truth of P_1 implies that P_2 is true, which implies that P_3 is true, which implies that P_4 is true, etc.

- Asks for justification and clarification from students for the Gauss' formula.
- Ask to students whether they have questions or not.
- Prove the Gauss' formula.
- Project the questions as follows.
- Then solve the question of the following on the board by using principles of mathematical induction:

a Prove that $1^2+2^2+3^2+4^2+.....+n^2=\frac{n(n+1)(2n+1)}{6}$ for all $n\in Z^+$.

b Find $1^2+2^2+3^2+4^2+.....+100^2$.

a P_n is: "$1^2+2^2+3^2+4^2+.....+n^2=\frac{n(n+1)(2n+1)}{6}$ for all $n \in Z^+$.

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=1^2=1$ and RHS $=\frac{1\times 2\times 3}{6}=1$

$\therefore$ P_1 is true

(2) If P_k is true, then

$1^2+2^2+3^2+4^2+.....+k^2=\frac{k(k+1)(2k+1)}{6}$ ..... $(*)$

Thus $1^2+2^2+3^2+4^2+.....+k^2+(k+1)^2$

$$=\frac{k(k+1)(2k+1)}{6}+(k+1)^2 \quad \{\text{using } *\}$$

$$=\frac{k(k+1)(2k+1)}{6}+(k+1)^2\times\frac{6}{6}$$

Note: Always look for common factors.

$$=\frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$

$$=\frac{(k+1)(2k^2+k+6k+6)}{6}$$

$$=\frac{(k+1)(2k^2+7k+6)}{6}$$

$$=\frac{(k+1)(k+2)(2k+3)}{6}$$

$$=\frac{(k+1)([k+1]+1)(2[k+1]+1)}{6}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore$ P_n is true {Principle of mathematical induction}

b $1^2+2^2+3^2+4^2+.....+100^2=\frac{100\times 101\times 201}{6}=338\,350$ {as $n=100$}

Transition: If you have no question, let's solve more problems.

D. Extend (15 min.)

- Distribute the worksheet.
- Students will try to solve questions about binomial distributions on the worksheet.
- Walk around and ask "how did you get this answers?"
- Check the students whether they solve the problems or not.
- answers will be checked on the board by writing the questions on the board

Transition: good job! Thank you, class. Have a nice day!

E. Evaluate (During the whole lesson):

- Assesses students' knowledge and skills through oral questions.
- Observe the students during the lesson.
- Take notes students' name if they have a problem when they solve questions.

9. Closure & Relevance for Future Learning

- Ask students to explain what they learn today.
- Then, Want students to write 3 key words that they have learned this lesson on their notebooks.
- Assign students to do the rest of the questions on the worksheet.

11. Modifications

- If students cannot remember previous lesson, give them some clues.

- If students do not give answer to your questions, wait 20 seconds more.
- Choose simple questions firstly to solve on the board.

a Prove that $1^2 + 2^2 + 3^2 + 4^2 + ... + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ for all $n \in Z^+$.

b Find $1^2 + 2^2 + 3^2 + 4^2 + ... + 100^2$.

TED ANKARA COLLEGE FOUNDATION HIGH SCHOOL WORKSHEET

Prove the following propositions, using the principle of mathematical induction:

1.

$$1 + 3 + 5 + 7 + ... + (2n - 1) = n^2, \quad n \in Z^+.$$

2.

$7^n + 2$ is divisible by 3, $n \in Z^+$.

3.

$$1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \ldots.. + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4},$$
$n \in Z^{+}$.

4.

$$1 + r + r^2 + r^3 + r^4 + \ldots.. + r^{n-1} = \frac{1 - r^n}{1 - r}, \quad n \in Z^{+}, \text{ provided that } r \neq 1.$$

5.

$3^n - 1 - 2n$ is always divisible by 4, for non-negative integers n.